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The Landau problem for nonvanishing functions with real coefficients

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Abstract

For the class of holomorphic functions $f(z) = f_0 + f_1 z + f_2 z^2 + \cdots$ in the unit disk $|z| < 1$ with $0 < |f(z)| < 1$ and $f_n \in \mathbb{R}$, $n = 0, 1, 2, \ldots$ we prove that $\max(f_0 + f_1 + \cdots + f_4) = \max|f_0 + f_1 + \cdots + f_4| = 1.46109\ldots$ and $\min(f_0 + f_1 + \cdots + f_4) = -0.713082\ldots$. © 2002 Elsevier Science B.V. All rights reserved.

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Let $\mathscr{H}(\mathbb{D})$ be the class of holomorphic functions in the unit disk $\mathbb{D} = \{z\colon |z| < 1\}$. We consider the following classes of functions:

$$\mathscr{B} = \{f \in \mathscr{H}(\mathbb{D})\colon f(z) = f_0 + f_1 z + f_2 z^2 + \cdots,\ |f(z)| < 1,\ z \in \mathbb{D}\},$$

$$\mathscr{B}_0 = \{f \in \mathscr{B}\colon |f(z)| > 0\},$$

$$\mathscr{W} = \{f \in \mathscr{B}\colon f(0) = 0\},$$

$$\mathscr{P} = \{p \in \mathscr{H}(\mathbb{D})\colon p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \Re p(z) > 0,\ z \in \mathbb{D}\},$$

$$\mathscr{P}_{\mathbb{R}} = \{p \in \mathscr{P}\colon p_n \in \mathbb{R},\ n = 1, 2, \ldots\}.$$

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With no loss of generality we may assume that for $f \in \mathscr{B}_0$ we have the form $f = e^{-tp}$ with $t > 0$ and $p \in \mathscr{P}$.

In [1] Landau proved that

$$\sup_{f \in \mathscr{B}} |\alpha_0 + \alpha_1 + \cdots + \alpha_n| = 1 + \sum_{k=1}^{n} \left(\frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{2 \cdot 4 \cdot \ldots \cdot 2k} \right)^2 := G_n, \quad n \in \mathbb{N},$$

with this result being sharp while $G_n \sim (1/\pi) \log n$, $n \to \infty$.

In [2] Lewandowski and Szynal consider the Landau problem for the class $\mathscr{B}_0$ and they prove that

$$\max_{f \in \mathscr{B}_0} |f_0 + f_1| \leqslant 2e^{-1/2} \simeq 1.21\ldots \quad \text{and} \quad \max_{f \in \mathscr{B}_0} |f_0 + f_1 + f_2| \simeq 1.33\ldots,$$

while in both cases the extremal functions are of the form $f = e^{-tp}$, with $t > 0$ and $p \in \mathscr{P}_{\mathbb{R}}$. They also conjecture that for any $f \in \mathscr{B}_0$ there exists a constant $L > 1$ and $n \in \mathbb{N}$ such that $\max_{f \in \mathscr{B}_0} |f_0 + f_1 + f_2 + \cdots + f_n| \leqslant L$.

In [3] Ganczar et al., consider the Landau problem for the class $\mathscr{B}_0(\mathbb{R})$ which consists of the functions $f \in \mathscr{B}_0$ with real Taylor coefficients and they prove that $S_3^* = 1.40315\ldots$ where

$$S_n^* = \max_{f \in \mathscr{B}_0(\mathbb{R})} |f_0 + f_1 + \cdots + f_n|, \quad (n = 2, 3, \ldots).$$

In this paper, we calculate S_4, S_4^* and s_4 where

$$S_n = \max_{f \in \mathscr{B}_0(\mathbb{R})} (f_0 + f_1 + \cdots + f_n),$$

and

$$s_n = \min_{f \in \mathscr{B}_0(\mathbb{R})} (f_0 + f_1 + \cdots + f_n), \quad (n = 2, 3, \ldots).$$

In the present paper, we face a problem involving the estimation of quantities which depend on the Taylor coefficients of functions belonging to the class $\mathscr{P}_{\mathbb{R}}$.

Our first idea is to use the Carathéodory–Toeplitz conditions as they are the strongest relations between the Taylor coefficients of the class $\mathscr{P}_{\mathbb{R}}$.

A second idea is to express these relations in such a way that each Taylor coefficient can be converted separately to a polynomial of several variables.

Combining these two ideas, we transform the initial problem into finding the max (or min) of a polynomial of several variables, defined in a closed interval $[0,1]^k$, $k \leqslant 4$.

A serious problem in this paper is the size of the polynomials which are involved in the elementary calculations. Using the computer algebra system Mathematica 4.0, we obtained all necessary results. We will need the following Lemmas.

Lemma 1. *Let $K_n(\mathscr{P}_{\mathbb{R}})$ be the set of $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for which there exists a $q(z) = 1 + q_1 z + q_2 z^2 + \cdots \in \mathscr{P}_{\mathbb{R}}$ having $q_1 = x_1$, $q_2 = x_2, \ldots, q_n = x_n$. Let also A_n be the set of $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$*

such that $D_k(x_1,x_2,\ldots,x_k)>0,\ k=1,2,\ldots,n$ *where*:

$$D_k(x_1,x_2,\ldots,x_k)=\begin{vmatrix} 2 & x_1 & x_2 & \ldots & x_k \\ x_1 & 2 & x_1 & \ldots & x_{k-1} \\ x_2 & x_1 & 2 & \ldots & x_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_k & x_{k-1} & x_{k-2} & \ldots & 2 \end{vmatrix}.$$

If $\bar{A}_n$ *is the closure of* A_n *then* $\bar{A}_n=K_n(\mathscr{P}_{\mathbb{R}})$.

The above Lemma is a part of the Carathéodory–Toeplitz Theorem (see [4,5]).

Lemma 2. *If* $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ $(n\leqslant 4)$ *the following propositions are equivalent.*

(i) $x\in K_n(\mathscr{P}_{\mathbb{R}})$

(ii) *there exists a* $(t_1,t_2,\ldots,t_n)\in[0,1]^n$ *such that*: $x_1=p_1(t_1)$, $x_2=p_2(t_1,t_2),\ldots,x_n=p_n(t_1,t_2,\ldots,t_n)$ *where*

$$p_1(t_1)=-2+4t_1,$$

$$p_2(t_1,t_2)=2+16t_1(-1+t_1+t_2-t_1t_2),$$

$$\begin{aligned}p_3(t_1,t_2,t_3)=&-2+4t_1(-3+4t_1)^2-32t_1t_2(1-5t_1+4t_1^2)-64t_1^2t_2^2(1+t_1)\\&+64(-1+t_1)t_1(1-t_2)t_2t_3,\end{aligned}$$

$$\begin{aligned}p_4(t_1,t_2,t_3,t_4)=2(1&+32t_1(-1+5t_1)+128t_1^3(-2+t_1)+32t_1t_2(1-9t_1+20t_1^2-12t_1^3)\\&+128t_1^2t_2^2(1-4t_1+3t_1^2)+128t_1^3t_2^3(1-t_1)+128t_1t_2t_3(1-3t_1+2t_1^2-t_2)\\&+128t_1^2t_2^2t_3(5-4t_1-2t_2+t_3+2t_1t_2-t_2t_3)+128t_1t_2^2t_3^2(-1+t_2)\\&+128t_1t_2t_3t_4(-1+t_1)(-1+t_2)(-1+t_3)).\end{aligned}$$

Proof. The quantity $D_k(x_1,x_2,\ldots,x_k)$ can be written as polynomial of second degree in x_k of the form:

$$-D_{k-2}(x_1,x_2,\ldots,x_{k-2})x_k^2+\cdots,\quad (D_k=1 \text{ for } k\leqslant 0).$$

If $\rho_k\equiv\rho_k(x_1,x_2,\ldots,x_{k-1})$ and $\rho_k^*\equiv\rho_k^*(x_1,x_2,\ldots,x_{k-1})$ are the roots of the above polynomial it is easy to see that the relation $x\in A_n$ is equivalent to $\min(\rho_k,\rho_k^*)<x_k<\max(\rho_k,\rho_k^*)$ or

$$x_k=\rho_k+t_k(\rho_k^*-\rho_k),\quad t_k\in(0,1),\quad k=1,2,\ldots,n. \tag{1}$$

- For $k=1$ we get $\rho_1=-2$ and $\rho_1^*=2$. Therefore

$$x_1=p_1(t_1),\quad t_1\in(0,1). \tag{2}$$

• For $k=2$ by the equation $D_2(x_1,x_2)=0$ we obtain $\rho_2=-2+x_1^2$ and $\rho_2^*=2$. Thus combining (1) with (2) we get

$$x_2 = p_2(t_1,t_2), \quad (t_1,t_2)\in(0,1)^2. \tag{3}$$

• For $k=3$ through equation $D_3(x_1,x_2,x_3)=0$ we obtain

$$\rho_3 = -\frac{4-2x_1-(x_1-x_2)^2}{-2+x_1} \quad \text{and} \quad \rho_3^* = -\frac{4+2x_1-(x_1+x_2)^2}{2+x_1}.$$

Consequently, combining (1) with (2) and (3) we obtain after the calculations

$$x_3 = p_3(t_1,t_2,t_3), \quad (t_1,t_2,t_3)\in(0,1)^3. \tag{4}$$

In the same manner we can see that $x_4=p_4(t_1,t_2,t_3,t_4)$. Summarizing, we have that the transform

$$(t_1,t_2,\ldots,t_n)\to(p_1(t_1),p_2(t_1,t_2),\ldots,p_n(t_1,t_2,\ldots,t_n))$$

is one-to-one from $(0,1)^n$ onto A_n. After the above observation the rest of the proof is straightforward. □

Lemma 3. *It holds that*

$$S_4=\max\{L_4(t,t_1,\ldots,t_4)\colon (t,t_1,\ldots,t_4)\in(0,+\infty)\times[0,1]^4\},$$

$$S_4^*=\max\{|L_4(t,t_1,\ldots,t_4)|\colon (t,t_1,\ldots,t_4)\in(0,+\infty)\times[0,1]^4\}$$

and

$$s_4=\min\{L_4(t,t_1,\ldots,t_4)\colon (t,t_1,\ldots,t_4)\in(0,+\infty)\times[0,1]^4\}$$

where

$$\begin{aligned} L_4(t,t_1,t_2,t_3,t_4) = \mathrm{e}^{-t}[&1-(p_1(t_1)+p_2(t_1,t_2)+p_3(t_1,t_2,t_3)+p_4(t_1,t_2,t_3,t_4))t \\ &+(\tfrac{1}{2}(p_1^2(t_1)+p_2^2(t_1,t_2))+p_1(t_1)(p_2(t_1,t_2)+p_3(t_1,t_2,t_3)))t^2 \\ &-\tfrac{1}{2}(\tfrac{1}{3}p_1^3(t_1)+p_1^2(t_1)p_2(t_1,t_2))t^3+\tfrac{1}{24}p_1^4(t_1)t^4]. \end{aligned}$$

Proof. Without loss of generality we suppose that $f\in\mathscr{B}_0(\mathbb{R})$ iff f can be written in the form $f=\mathrm{e}^{-tp}$ with $t>0$ and $p\in\mathscr{P}_{\mathbb{R}}$. Therefore from Lemma 2 it follows that

$$f(z)=\mathrm{e}^{-t[1+p_1(t_1)z+p_2(t_1,t_2)z^2+\cdots+p_n(t_1,t_2,\ldots,t_n)z^n+\cdots]}. \tag{5}$$

By Taylor expansion of the previous form of f after elementary calculations we obtain $f_0+f_1+\cdots+f_3=L_3$ and $f_0+f_1+\cdots+f_4=L_4$. □

For the next Theorem we will need the following polynomials:

$$\begin{aligned} \varphi_0(t) = {}&-441364956+2205492580t-4742725278t^2+5815050924t^3-4530686063t^4 \\ &+2361706809t^5-842664813t^6+206357288t^7-34085280t^8+3627264t^9 \\ &-224512t^{10}+6144t^{11}, \end{aligned} \tag{6}$$

$$\Phi_1(t) = 636264 + 5357284t - 17556360t^2 + 21082900t^3 - 13386386t^4 + 5017897t^5 - 1148066t^6 + 157608t^7 - 11936t^8 + 384t^9, \tag{7}$$

$$\Psi_1(t) = 4t(4281296 - 11410068t + 12647744t^2 - 7631800t^3 + 2752352t^4 - 610423t^5 + 81708t^6 - 6064t^7 + 192t^8), \tag{8}$$

$$\Phi_2(t,t_1) = -11 - 4t^2(1-2t_1)^2 - 16t_1(-7+8t_1) + 4t(5 - 30t_1 + 32t_1^2), \tag{9}$$

$$\Psi_2(t,t_1) = 64(-2+t)(-1+t_1)t_1, \tag{10}$$

$$\Phi_3(t,t_1,t_2) = 1 + 2t + 4t_1(2 - t - 2t_2), \tag{11}$$

$$\Psi_3(t,t_1,t_2) = 8(1-t_2), \tag{12}$$

$$\varphi_0^*(t) = 3(-17+4t)^2(-575347 + 1291208t - 1081440t^2 + 431104t^3 - 82688t^4 + 6144t^5), \tag{13}$$

$$\Phi_1^*(t) = -5073 - 9478t + 11720t^2 - 3808t^3 + 384t^4, \tag{14}$$

$$\Psi_1^*(t) = 4t(-7529 + 6796t - 2000t^2 + 192t^3), \tag{15}$$

$$\Phi_2^*(t,t_1) = -3 + 4t(1-t) - 8tt_1(7 - 2t - 8t_1 + 2tt_1), \tag{16}$$

$$\Psi_2^*(t,t_1) = 64t(-1+t_1)t_1, \tag{17}$$

$$\Phi_3^*(t,t_1,t_2) = 3 - 2t + 4t_1(-2 + t + 2t_2) \quad \text{and} \tag{18}$$

$$\Psi_3^*(t,t_1,t_2) = 8t_2. \tag{19}$$

Considering the Landau problem for $\mathscr{B}_0(\mathbb{R})$ we find the following partial results:

Theorem 1. *For S_4, S_4^* and s_4 it holds that*:

(i) *$S_4 = 1.46109\ldots$ is the value of the function $L_4(t,t_1,t_2,t_3,1)$ for $t = t_0 \simeq 0.780114\ldots$, while t_0 is the root of the equation $\varphi_0(t) = 0$,*

$$t_1 = \frac{\Phi_1(t)}{\Psi_1(t)}, \qquad t_2 = \frac{\Phi_2(t,t_1)}{\Psi_2(t,t_1)} \quad \textit{and} \quad t_3 = \frac{\Phi_3(t,t_1,t_2)}{\Psi_3(t,t_1,t_2)},$$

(ii) *$S_4^* = S_4$ and*

(iii) *$s_4 = -0.713082\ldots$ is the value of the function $L_4(t,t_1,t_2,t_3,0)$ for $t = t_0^* \simeq 1.11167\ldots$, while t_0^* is the root of the equation $\varphi_0^*(t) = 0$,*

$$t_1 = \frac{\Phi_1^*(t)}{\Psi_1^*(t)}, \qquad t_2 = \frac{\Phi_2^*(t,t_1)}{\Psi_2^*(t,t_1)} \quad \textit{and} \quad t_3 = \frac{\Phi_3^*(t,t_1,t_2)}{\Psi_3^*(t,t_1,t_2)}.$$

Proof. At first, we remark that:

1. If we set in any function L_k ($k=1,\ldots,4$), $t_i=0$ or $t_i=1$ ($i=1,2,3,4$), the value of the function does not depend on the variables t_j when $j>i$.

2. The polynomial L_4 is of first degree in t_4 with corresponding coefficient $-256e^{-t}t(-1+t_1)t_1(-1+t_2)t_2(-1+t_3)t_3$ which is non-negative. Therefore in order to find S_4 we set $t_4=1$ and for s_4 we set $t_4=0$.

3. In order to calculate value S_4 we consider all the restrictions of the function L_4 for $t_4=1$, $t_i=1$ and $t_i=0$, $i=1,2,3$. We then find all critical points in the interior of the definition domains of each restriction and their corresponding values. The larger of these values coincides with S_4. In a similar way we are working for s_4.

4. Let $\alpha=(\alpha_0,\alpha_1,\ldots,\alpha_n)\in\mathbb{C}^n$, $b=(b_0,b_1,\ldots,b_k)\in\mathbb{C}^k$, $P_\alpha(x)=\alpha_0+\alpha_1x+\ldots,\alpha_nx^n$ and $P_b(x)=b_0+b_1x+\cdots+b_kx^k$. The equations $P_\alpha(x)=0$ and $P_b(x)=0$ yield two relations of the form $A(\alpha,b)x+B(\alpha,b)=0$ and $C(\alpha,b)=0$ where $A(\alpha,b)$, $B(\alpha,b)$ and $C(\alpha,b)$ are polynomials in each element of α, b, with the following procedure. The first polynomial $P_\alpha(x)$ is divided with the second $P_b(x)$, proceeding with the division of $P_b(x)$ with the remainder of the initial polynomial division. The final elimination of x is achieved by recursions of the procedure described, which will be used several times through this paper.

- For $t_4=1$:

Equation $\partial L_4/\partial t_3=0$ gives

$$h_1(t,t_1,t_2)(-1-2t-8t_1+4tt_1+8t_1t_2+8t_3-8t_2t_3)=0, \tag{20}$$

where $h_1(t,t_1,t_2)=-e^{-t}64t(-1+t_1)t_1(-1+t_2)t_2$. Since $h_1(t,t_1,t_2)\neq 0$, then

$$t_3\Psi_3(t,t_1,t_2)-\Phi_3(t,t_1,t_2)=0. \tag{21}$$

Eliminating t_3 between (21) and $\partial L_4/\partial t_2=0$ we get

$$\begin{aligned}&h_2(t,t_1)(11-20t+4t^2-112t_1+120tt_1-16t^2t_1+128t_1^2-128tt_1^2+16t^2t_1^2+128t_1t_2\\&\quad-64tt_1t_2-128t_1^2t_2+64tt_1^2t_2)=0\end{aligned} \tag{22}$$

where $h_2(t,t_1)=e^{-t}4t(-1+t_1)t_1$. Since $h_2(t,t_1)\neq 0$, then

$$t_2\Psi_2(t,t_1)-\Phi_2(t,t_1)=0. \tag{23}$$

Eliminating t_2 between $\partial L_4/\partial t_1=0$, $\partial L_4/\partial t=0$ and (23) we have, respectively,

$$\frac{h_3(t,t_1)}{(-2+t)^2}=0, \tag{24}$$

where

$$\begin{aligned}h_3(t,t_1)=&(-342+2550t-5331t^2+3592t^3-888t^4+16t^6-16t_1(36+504t-1431t^2+799t^3\\&-74t^4-52t^5+8t^6)+96t_1^2(128-318t+20t^2+103t^3-44t^4+4t^5)-256t^2t_1^3\\&(-12-112t+93t^2-27t^3+2t^4)-256t^3t_1^4(64+50t-14t^2+t^3))\end{aligned}$$

and

$$\frac{h_4(t,t_1)}{(-2+t)}=0, \tag{25}$$

where

$$h_4(t,t_1) = t(-18 - 135t + 174t^2 + 4t^3 - 8t^4 - 12tt_1(-32 + 37t + 16t^2) + t^2t_1^2(-2 - 15t + 2t^2) + 8t^3t_1^3(-64 + 8t)).$$

Applying the procedure described in Remark 4 to (24) and (25) with respect to t_1 we get

$$\frac{h_5(t,t_1)}{(-8+t)^2(-2+t)^2} = 0, \tag{26}$$

where

$$\begin{aligned} h_5(t,t_1) = {} & -6516 + 9224t - 17268t^2 + 21356t^3 - 11787t^4 + 3048t^5 - 364t^6 + 16t^7 \\ & - 8tt_1(-2152 - 4968t + 10698t^2 - 6314t^3 + 1605t^4 - 186t^5 + 8t^6) \\ & + 16t^2t_1^2(-4548 + 6736t - 3560t^2 + 852t^3 - 95t^4 + 4t^5)) \end{aligned}$$

and

$$(-8+t)^2(-2+t)[t_1\Psi_1(t) - \Phi_1(t)] = 0. \tag{27}$$

Therefore for $t \neq 2$ and $t \neq 8$ by (26) and (27) we obtain

$$-3(-4548 + 6736t - 3560t^2 + 852t^3 - 95t^4 + 4t^5)^2\varphi_0(t) = 0. \tag{28}$$

Solving Eq. (28) with Mathematica 4.0 we succeed in finding all its roots which are: $\rho_1 = \rho_2 = 1.54984\ldots$, $\rho_3 = \rho_4 = 2.63304\ldots$, $\rho_5 = \rho_6 = 8.31271\ldots$, $\rho_7 = \rho_8 = 5.6272\ldots - i\ 1.36096\ldots$, $\rho_9 = \rho_{10} = 5.6272\ldots + i\ 1.36096\ldots$, $\rho_{11} = 0.780114\ldots$, $\rho_{12} = 1.20285\ldots$, $\rho_{13} = 1.67847\ldots$, $\rho_{14} = 1.81244\ldots$, $\rho_{15} = 2.3582\ldots$, $\rho_{16} = 2.98161\ldots$, $\rho_{17} = 4.12162\ldots$, $\rho_{18} = 4.46949\ldots - i\ 0.815353\ldots$, $\rho_{19} = 4.46949\ldots + i\ 0.815353\ldots$, $\rho_{20} = 6.33369\ldots - i\ 1.39775\ldots$, $\rho_{21} = 6.33369\ldots + i\ 1.39775\ldots$.

- For $t = 2$:

the equation $h_5(2,t_1) = 0$ has the roots $r_1 = (4-\sqrt{3})/8$ and $r_2 = (4+\sqrt{3})/8$. Solving the system of equations $h_5(2,t_1) = 0$, $\partial L_4/\partial t_1 = 0$ and (21) with respect to t_1, t_2 and t_3 we obtain as solutions the points

$$\begin{aligned} (t_1,t_2,t_3) &= \left(\frac{4-\sqrt{3}}{8}, \frac{-5+6\sqrt{3}}{13\sqrt{3}}, -\frac{54+19\sqrt{3}}{4(-3+\sqrt{3})(13+6\sqrt{3})}\right) \\ &\simeq (0.283494\ldots, 0.239481\ldots, 0.732537\ldots) \end{aligned}$$

and

$$\begin{aligned} (t_1,t_2,t_3) &= \left(\frac{4+\sqrt{3}}{8}, \frac{5+6\sqrt{3}}{13\sqrt{3}}, \frac{-54+19\sqrt{3}}{4(3+\sqrt{3})(-13+6\sqrt{3})}\right) \\ &\simeq (0.716506\ldots, 0.683596\ldots, 0.427299\ldots). \end{aligned}$$

For $t = 8$:
in the same manner we obtain the point

$$(t_1, t_2, t_3) = \left(\frac{42 - \sqrt{615}}{32}, -\frac{1387 - 40\sqrt{615}}{3(-42 + \sqrt{615})(-10 + \sqrt{615})}, \frac{(-42 + \sqrt{615})(-184 + 7\sqrt{615})}{4(-1123 + 49\sqrt{615})} \right)$$

$$\simeq (0.537525\ldots, 0.517279\ldots, 0.485528\ldots).$$

Using the roots of Eq. (28) and the solutions of the systems formed in the cases $t = 2$ and $t = 8$, after the calculations we find that all the critical points of function $L_4(t, t_1, t_2, t_3, 1)$ for $(t, t_1, t_2, t_3) \in (0, \infty) \times (0, 1)^3$, belong to the set

$$\begin{aligned} A_1 = \{&(1.54984\ldots, 0.595751\ldots, 0.079176\ldots, 0.650919\ldots),\ (0.780114\ldots, 0.407469\ldots,\\ &0.389277\ldots, 0.671242\ldots),\ (1.20285\ldots, 0.244965\ldots, 0.549669\ldots, 0.863142\ldots),\\ &(1.67847\ldots, 0.386081\ldots, 0.0835104\ldots, 0.626787\ldots),\ (1.81244\ldots, 0.758132\ldots,\\ &0.699756\ldots, 0.395305\ldots),\ (2.3582\ldots, 0.597436\ldots, 0.867069\ldots, 0.673483\ldots),\\ &(2.98161\ldots, 0.171205\ldots, 0.156065\ldots, 0.900136\ldots),\ (4.12162\ldots, 0.764237\ldots,\\ &0.268454\ldots, 0.190784\ldots),\ (8.31271\ldots, 0.534529\ldots, 0.516905\ldots, 0.496206\ldots),\\ &(2.000000\ldots, 0.283494\ldots, 0.239481\ldots, 0.732537\ldots),\ (2.000000\ldots, 0.716506\ldots,\\ &0.683596\ldots, 0.427299\ldots),\ (8.000000\ldots, 0.537525\ldots, 0.517279\ldots, 0.485528\ldots)\}, \end{aligned}$$

$\max\{L_4(t, t_1, t_2, t_3, 1) : (t, t_1, t_2, t_3) \in A_1\} = 1.461094\ldots$ and $\min\{L_4(t, t_1, t_2, t_3, 1) : (t, t_1, t_2, t_3) \in A_1\} = 0.004103\ldots$.

- For $t_4 = 0$:

proceeding exactly as in previous case, we derive the set

$$\begin{aligned} A_2 = \{&(1.11167\ldots, 0.594775\ldots, 0.617521\ldots, 0.324184\ldots),\ (2.36682\ldots, 0.300531\ldots,\\ &0.730184\ldots, 0.0792381\ldots),\ (2.76576\ldots, 0.80512\ldots, 0.478528\ldots, 0.788036\ldots)\}, \end{aligned}$$

$\max\{L_4(t, t_1, t_2, t_3, 0): (t, t_1, t_2, t_3) \in A_2\} = -0.3034\ldots$ and $\min\{L_4(t, t_1, t_2, t_3, 0) : (t, t_1, t_2, t_3) \in A_2\} = -0.713082\ldots$.

real values for t_1, t_2 and t_3.

- For $t_3 = 1$:

we consider the equation

$$\frac{\partial L_4}{\partial t_2} = \frac{\partial L_4}{\partial t_1} = \frac{\partial L_4}{\partial t} = 0. \tag{29}$$

Applying to (29) the procedure of eliminating t_2 and t_1, we examine separately as we did previously, some cases of the form $q_i(t) = 0$, $i = 1, 2, 3, \ldots$. Thus we obtain three polynomial equations of the form

$$t_2 \Psi_2^{**}(t, t_1) - \Phi_2^{**}(t, t_1) = 0, \tag{30}$$

$$t_1 \Psi_1^{**}(t) - \Phi_1^{**}(t) = 0 \tag{31}$$

and

$$\varphi_0^{**}(t) = 0. \tag{32}$$

We omit giving explicitly the polynomials $q_i(t)$, $\Phi_1^{**}(t)$, $\Phi_2^{**}(t,t_1)$, $\Psi_1^{**}(t)$, $\Psi_2^{**}(t,t_1)$, and $\varphi_0^{**}(t)$ because of their size. Especially the last polynomial $\varphi_0^{**}(t)$ can be converted to a product of factors of the form

$$9(-1+2t)\varphi_1^2(t)\varphi_2^2(t)\varphi_3(t)\varphi_4^2(t), \tag{33}$$

where $\varphi_1(t)$ is of sixth degree, $\varphi_2(t)$ of twelfth degree, $\varphi_3(t)$ of fifteenth degree and $\varphi_4(t)$ of thirtieth degree with respect to t, respectively. Continuing we find all one hundred and twelve roots of Eq. (32).

Using the roots of Eq. (32) and the points that satisfy the relations $q_i(t) = 0$ $i = 1,2,3,\ldots$, after the calculations we find that all the critical points of function $L_4(t,t_1,t_2,1,1)$ for $(t,t_1,t_2) \in (0,\infty) \times (0,1)^2$, belong to the set

$$\begin{aligned}
A_3 = \{&(0.500000\ldots, 0.499999\ldots, 0.500000\ldots),\ (0.982144\ldots, 0.278639\ldots, 0.423554\ldots),\\
&(0.563644\ldots, 0.244881\ldots, 0.173675\ldots),\ (1.141401\ldots, 0.133612\ldots, 0.277829\ldots),\\
&(1.385251\ldots, 0.200877\ldots, 0.627579\ldots),\ (1.658795\ldots, 0.059786\ldots, 0.077125\ldots),\\
&(3.340424\ldots, 0.124864\ldots, 0.185539\ldots),\ (0.398400\ldots, 0.264699\ldots, 0.180761\ldots),\\
&(0.982222\ldots, 0.415880\ldots, 0.309214\ldots),\ (1.146923\ldots, 0.207432\ldots, 0.632673\ldots),\\
&(1.499987\ldots, 0.215201\ldots, 0.502770\ldots),\ (2.407662\ldots, 0.192559\ldots, 0.338093\ldots),\\
&(2.690036\ldots, 0.181608\ldots, 0.254821\ldots),\ (3.94974\ldots, 0.468006\ldots, 0.793194\ldots),\\
&(1.07394\ldots, 0.175938\ldots, 0.0801832\ldots),\ (1.07394\ldots, 0.214171\ldots, 0.588422\ldots),\\
&(6.28933\ldots, 0.451604\ldots, 0.702356\ldots),\ (0.976822\ldots, 0.159528\ldots, 0.356847\ldots),\\
&(0.987624\ldots, 0.0516469\ldots, 0.446299\ldots),\ (2.55783\ldots, 0.185094\ldots, 0.306034\ldots),\\
&(10.243\ldots, 0.458744\ldots, 0.639068\ldots),\ (47.9003\ldots, 0.441665\ldots, 0.935745\ldots),\\
&(47.9003\ldots, 0.510175\ldots, 0.508884\ldots),\ (0.3984\ldots, 0.264699\ldots, 0.180761\ldots),\\
&(0.982066\ldots, 0.094002\ldots, 0.426308\ldots),\ (1.14692\ldots, 0.216318\ldots, 0.552922\ldots),\\
&(1.49999\ldots, 0.207221\ldots, 0.549478\ldots),\ (2.40766\ldots, 0.199474\ldots, 0.301101\ldots),\\
&(2.69004\ldots, 0.191379\ldots, 0.201498\ldots),\ (0.982144\ldots, 0.00346203\ldots, 0.467665\ldots),\\
&(0.982144\ldots, 0.105913\ldots, 0.418354\ldots)\},
\end{aligned}$$

$\max\{L_4(t,t_1,t_2,1,1) : (t,t_1,t_2) \in A_3\} = 1.420327\ldots$ and $\min\{L_4(t,t_1,t_2,1,1) : (t,t_1,t_2) \in A_3\} = -10^{-18}$ $2.25246\ldots$.

• For $t_3 = 0$:
proceeding exactly as in previous case we derive

$$\begin{aligned} A_4 = \{&(0.637970\ldots, 0.801068\ldots, 0.905473\ldots),\ (2.766761\ldots, 0.460642\ldots, 0.710075\ldots),\\ &(4.271359\ldots, 0.806114\ldots, 0.934007\ldots),\ (0.979406\ldots, 0.736749\ldots, 0.403476\ldots),\\ &(0.807417\ldots, 0.586126\ldots, 0.814603\ldots),\ (0.840473\ldots, 0.759806\ldots, 0.391577\ldots),\\ &(1.236800\ldots, 0.377464\ldots, 0.958084\ldots),\ (4.656304\ldots, 0.8333266\ldots, 0.428365\ldots),\\ &(1.526060\ldots, 0.8892278\ldots, 0.792855\ldots),\ (1.921988\ldots, 0.916280\ldots, 0.494162\ldots),\\ &(3.440666\ldots, 0.343814\ldots, 0.766327\ldots),\ (3.71729\ldots, 0.754259\ldots, 0.203692\ldots),\\ &(0.979406\ldots, 0.457355\ldots, 0.843811\ldots),\ (0.979406\ldots, 0.740337\ldots, 0.38198\ldots),\\ &(8.10891\ldots, 0.625988\ldots, 0.669617\ldots),\ (1.67665\ldots, 0.380592\ldots, 0.633951\ldots),\\ &(11.7856\ldots, 0.588532\ldots, 0.625032\ldots),\ (49.2188\ldots, 0.500368\ldots, 0.508855\ldots),\\ &(49.2188\ldots, 0.567542\ldots, 0.928609\ldots),\ (1.52606\ldots, 0.292172\ldots, 0.0742202\ldots),\\ &(1.58586\ldots, 0.641036\ldots, 0.362164\ldots),\ (4.27136\ldots, 0.658569\ldots, 0.613046\ldots)\}, \end{aligned}$$

$\max\{L_4(t, t_1, t_2, 0, 1)\colon (t, t_1, t_2) \in A_4\} = 0.838021\ldots$ and $\min\{L_4(t, t_1, t_2, 0, 1)\colon (t, t_1, t_2) \in A_4\} = -0.622477\ldots$.

• For $t_2 = 1$:
by the same procedure, through simpler calculations than before we have

$$\begin{aligned} A_5 = \{&(0.593731\ldots, 0.713594\ldots),\ (1.3839\ldots, 0.333669\ldots),\ (1.66779\ldots, 0.885304\ldots),\\ &(2.89925\ldots, 0.603695\ldots),\ (3.13721\ldots, 0.193606\ldots),\ (3.81812\ldots, 0.924642\ldots),\\ &(3.250000\ldots, 0.307692\ldots)\}, \end{aligned}$$

$\max\{L_4(t, t_1, 1, 1, 1)\colon (t, t_1) \in A_5\} = 0.471131\ldots$ and $\min\{L_4(t, t_1, 1, 1, 1)\colon (t, t_1) \in A_5\} = -0.604323\ldots$.

• For $t_2 = 0$:
in the same manner we get

$$\begin{aligned} A_6 = \{&(0.327828\ldots, 0.759498\ldots),\ (0.329728\ldots, 0.40951\ldots),\ (0.355914\ldots, 0.097935\ldots),\\ &(0.738627\ldots, 0.016\ldots),\ (0.874242\ldots, 0.595843\ldots),\ (0.907488\ldots, 0.215898\ldots),\\ &(1.23589\ldots, 0.973422\ldots),\ (2.04847\ldots, 0.376249\ldots),\ (3.65511\ldots, 0.126236\ldots),\\ &(5.16722\ldots, 0.802415\ldots),\ (0.333617\ldots, 0.514823\ldots),\ (0.806102\ldots, 0.49065\ldots),\\ &(1.14202\ldots, 0.232493\ldots),\ (3.61717\ldots, 0.133629\ldots),\ (8.07298\ldots, 0.54914\ldots),\\ &(0.935822\ldots, 0.192564\ldots),\ (0.935822\ldots, 0.541392\ldots),\ (3.30541\ldots, 0.18893\ldots),\\ &(7.75877\ldots, 0.562449\ldots)\}, \end{aligned}$$

$\max\{L_4(t,t_1,0,1,1)\colon (t,t_1) \in A_6\} = 1.34838\ldots$ and $\min\{L_4(t,t_1,0,1,1)\colon (t,t_1) \in A_6\} = 0.036987\ldots$.

• For $t_1 = 1$:
the equation $\partial L_4/\partial t = 0$ yields

$$-\tfrac{1}{3}\mathrm{e}^{-t}(27 - 96t + 84t^2 - 24t^3 + 2t^4) = 0, \tag{34}$$

therefore

$A_7 = \{0.414236\ldots, 1.44592\ldots, 3.29095\ldots, 6.8489\ldots\}$, $\max\{L_4(t,1,1,1,1)\colon t \in A_7\} = 0.309099\ldots$ and $\min\{L_4(t,1,1,1,1)\colon t \in A_7\} = -0.405924\ldots$.

• For $t_1 = 0$:
similarly we get

$$-\tfrac{1}{3}\mathrm{e}^{-t}(3 - 24t + 36t^2 - 16t^3 + 2t^4) = 0, \tag{35}$$

with $A_8 = \{0.161274\ldots, 0.872881\ldots, 2.26831\ldots, 4.69754\ldots\}$, $\max\{L_4(t,0,1,1,1)\colon t \in A_8\} = 1.253427\ldots$ and $\min\{L_4(t,0,1,1,1)\colon t \in A_8\} = 0.838993\ldots$.

Comparing the occurring values for the maximum and the minimum of L_4 of the relative sets A_i $(i = 1,\ldots,8)$, we obtain the result needed. $\square$

Remark. If we had considered initially the class $\mathscr{B}_4^* = \{f \in \mathscr{B}_0\colon f_i \in \mathbb{R},\ i = 1,\ldots,4\}$ instead of the class $\mathscr{B}_0(\mathbb{R})$, all our evaluations wouldn't have been modified.

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